Reconstruction of time-dependent boundary heat flux by a BEM-based inverse algorithm

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Abstract

The objective of this study is to reconstruct an unknown time-dependent heat flux distribution at a surface whose temperature history is provided by a broad-band thermochromic liquid crystal (TLC) thermographic technique. The information given for this inverse problem is the surface temperature history. Although this is not an inverse problem, it is solved as such in order to filter the errors in input temperatures which are reflected in errors in heat fluxes. We minimize a quadratic functional which measures the sum of the squares of the deviation of estimated (computed) temperatures relative to measured temperatures provided by the TLC thermography. The objective function is minimized using the Levenberg–Marquardt method, and we develop an explicit scheme to compute the required sensitivity coefficients. The unknown flux is allowed to vary in space and time. Results are presented for a simulation in which a spatially varying and time-dependent flux is reconstructed over an airfoil.

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1. Introduction

The subject of this study is the retrieval of multi-dimensional surface heat fluxes using temperatures measured at the surface of a body with thermochromic liquid crystals (TLCs). TLCs are applied to a surface which is subjected to an unknown heat flux or convective boundary condition, and they either respond by changing color over a relatively large range of temperatures (around 20°C) in the case of broad-band TLC \cite{1,2,3} or at a characteristic temperature, $T_c$, in the case of narrow-band TLC \cite{4}. Standard practice is to employ the solution of a semi-infinite medium initially at a uniform temperature and subjected to an impulsive convective boundary condition in order to retrieve unknown convective heat coefficients. This approach obviously neglects multi-dimensional effects and forces experimentalists to construct tests which conform to the 1D. Consequently, experiments often do not capture the very physics of the heat transfer process they set out to investigate. Recently, inverse problem methods and the Boundary Element Method (BEM) have been proposed to resolve multi-dimensional effects in retrieving unknown heat transfer coefficients using broad-band TLC \cite{3}.

In this paper, we further develop the inverse-based BEM approach to reconstruction of unknown multi-dimensional and time-dependent heat fluxes. Ultimately, a heat transfer coefficient can be evaluated once the heat flux is known by invoking Newton’s cooling law, and it is this quantity which is of interest to designers of thermal systems. The information given for this inverse problem is the surface temperature history measured by TLC at a surface exposed to convective heat transfer. Although this is not an inverse problem, it is solved as such in order to filter the errors in input temperatures which are reflected in errors in heat fluxes. We minimize a quadratic functional which measures the sum of the squares of the deviation of estimated (computed) temperatures relative to those measured temperatures provided by the TLC thermography. The objective function is minimized using the Levenberg–Marquardt (LM) method.

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2. The BEM solution of the heat conduction problem

In the case of irregular domain geometry, say a turbine blade, a combustor wall, etc., the analytical solution for the field problem which is required at each step of the inverse problem of interest is impossible. In such cases, numerical methods must be adopted, and we choose to use the BEM for this purpose. The BEM only requires a surface mesh, and as such, it is very flexible and readily applicable to complex geometries without the intricate mesh generation required by traditional finite difference method (FDM) and finite element method (FEM). Furthermore, the unknowns which appear in the BEM are the boundary temperature and fluxes, precisely the variables of interest in the problem under consideration. Boundary fluxes are computed as part of the solution and not by post-processing numerical differentiation. Further, as unknowns appear on the boundary, no unnecessary internal information is generated, as would be the case in FDM- and FEM-based approaches. There are three common BEM formulations for transient heat conduction problems: the Laplace transform method [5], the dual reciprocity BEM [6], and the time-dependent fundamental solution method [7]. The latter is briefly reviewed here for completeness and in order to point out certain features which are relevant to the problem. In the BEM, the diffusion equation is first converted to a boundary integral equation (BIE) by: (1) multiplying the governing equation by a test function \( G(x, t, \xi, t_F) \), (2) integrating over the spatial domain and over time, (3) integrating by parts once in time and twice in space using Green’s first identity, and (4) invoking properties of the Green free-space solution identified as the test function \( G(x, t, \xi, t_F) \), resulting in

\[
\rho c C(\xi) T(\xi, t_F) = \int_{t_0}^{t_f} \int_I \left[ H(r, t, \xi, t_F) T(r, t) - q(r, t) G(r, t, \xi, t_F) \right] d\Gamma(r) dt. 
\]  

(1)

This BIE is valid for boundary or interior points. Here, the temperature is \( T(x, t) \), the independent variable \( t \) depends both on the spatial dimension of the problem and coordinate system of choice, while \( t \) denotes time. Thermophysical properties, which are the thermal conductivity \( k \), density \( \rho \), and specific heat \( c \), are taken as constants. \( \Gamma \) is the domain boundary, \( H(r, t, \xi, t_F) = -k \frac{\partial G(r, t, \xi, t_F)}{\partial n} \), \( q(r, t) = -k \frac{\partial T(r, t)}{\partial n} \), and \( \partial/\partial n \) denotes the normal derivative with respect to the outward-drawn normal. The free term, \( C(\xi) \) is 1, for \( \xi \in \Omega \) and is equal to the internal angle subtended at a point on the boundary, \( \xi \in \Gamma \), divided by \( 2\pi \) radians in 2D and \( 4\pi \) steradians in 3D. The Green free-space solution for the diffusion equation which solves the adjoint diffusion equation perturbed in free space by a Dirac delta function located at \( r = \xi \) and \( t = t_F \) is

\[
G(r, t, \xi, t_F) = \left[ \frac{-\rho c}{4\pi k(t_F - t)} \right]^{1/2} \exp \left[ \frac{-\rho c}{4k(t_F - t)} (r - \xi)^2 \right]. 
\]  

(2)

where \( d = 1, 2, 3 \) for 1D, 2D, and 3D problems, respectively, and \( r \) is the Euclidean distance from the field point \( r \) to the source point \( \xi \). In Eq. (1), the initial condition \( T(r, t_0) \) is taken to be either zero or a constant, and consequently, the domain integral arising from the initial condition is either zero or can be eliminated, and as such does not appear explicitly. The boundary, \( \Gamma \) is discretized using \( N \) boundary elements, and assuming \( F \) constant in time elements,

\[
\Gamma = \sum_{j=1}^{NE} \Gamma_j \quad \text{and} \quad (t_F - t_0) = \sum_{p=1}^{F} (t_p - t_{p-1}), 
\]  

(3)

where \( t_F \) is the final time and \( t_0 \) is the initial time, a convolution BEM time marching scheme is developed, see Brebbia et al. [7],

\[
\rho c C_i T^F_i = \sum_{p=1}^{F} \sum_{j=1}^{NE} \left( T^F_j H^F_{ij} - q^F_j G^F_{ij} \right) 
\]  

(4)

and the time-dependent influence coefficients \( H^F_{ij} \) and \( G^F_{ij} \) are,

\[
H^F_{ij} = \int_{t_{p-1}}^{t_p} \int_{\Gamma_j} H(r, t, \xi, t_F) d\Gamma(r) dt, 
\]  

(5)

\[
G^F_{ij} = \int_{t_{p-1}}^{t_p} \int_{\Gamma_j} G(r, t, \xi, t_F) d\Gamma(r) dt. 
\]  

The discretized BIE relates the temperature at any collocation point \( \xi_i \) (either on the boundary or in the domain) and any collocation time \( t_F \) with boundary temperatures and heat fluxes at current and previous time levels. Collocating at \( \xi_i \) boundary nodes \( (i = 1, \ldots, N) \), and setting the current time level to \( t_F \), the above is re-arranged as

\[
[H^F]^T \{ T \}^F - \{ G \}^T \{ q \}^F = \sum_{p=1}^{F-1} \{ G \}^T \{ q \}^F - \{ H \}^T \{ T \}^F, 
\]  

(6)

where

\[
[H]^F = [H]^F - \rho C[T]^F \quad \text{with} \quad [T] \quad \text{denoting the identity matrix}. 
\]  

The right-hand side contains convolution information for the problem and can be simply collapsed into a vector \( [c]^F \), while the left-hand side can be reordered according to boundary conditions to obtain a final simple algebraic form, \( [A]^F [x]^F = [b]^F \), which is solved at every time level \( F \) by direct or iterative methods to obtain unknown boundary values.
3. Objective function

In the process of solving the inverse problem, the heat flux is approximated using shape functions in space \( N_i(r) \) and shape functions in time \( N_j(t) \) as (Fig. 1)

\[
q(r, t) \approx \sum_{f=1}^{F} \sum_{i=1}^{N} N_j(t) N_i(r) q_f^j.
\]

(7)

We may assume, for example, \( N_j(t) \) to be piecewise unit constant shape functions over the time approximating interval \( t_f - t_{f-1} \), which need not correspond to, and in general does not correspond, to the BEM time stepping interval, \( t_p - t_{p-1} \).

\[
N^j(t) = \begin{cases} 1 & \text{if } 0 < t \leq t^j, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
N^f(t) = \begin{cases} 1 & \text{if } t^{f-1} < t \leq t^f, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
N^{f^1}(t) = \begin{cases} 1 & \text{if } t^{f^1-1} < t \leq t^{f^1}, \\ 0 & \text{otherwise}. \end{cases}
\]

(8)

Any other type of temporal shape functions may be used, for instance, linear. However, in this paper, we assume the constant shape functions given in Eq. (8). In space however, we assume linear roof-like shape functions as illustrated in Fig. 2, where the heat flux is interpolated at the right wall of a square region at the time approximating level \( f \).

At any time level \( f \), if \( N = M \), then \( \bar{q}_f^j \) is the computed flux at the measured surface along each of \( j = 1, 2 \ldots M \) measuring points averaged over the \( p \)-measuring time levels spanned by the approximating time interval \( \Delta t_f = t_f - t_{f-1} \). When \( N < M \), then \( \bar{q}_f^j \) is interpreted as a knot value for the functional approximation of the flux, and the value of the flux at the measuring point is spatially interpolated using Eq. (7). The collection \( \{ \bar{q}_f^j \} \) is denoted by the vector \( \bar{q} \) of length \([N \times F] \) which is denoted by the column vector \( \bar{q} = [\bar{q}_1^1, \bar{q}_2^1, \ldots, \bar{q}_N^1, \bar{q}_1^2, \bar{q}_2^2, \ldots, \bar{q}_N^2, \ldots, \bar{q}_N^F] \). For the current estimated values of the vector \( \bar{q} \), a quadratic function is defined to measure the difference between the temperature predicted at each node by the BEM code \( T^p_{j,m} \) and the measured temperature at each of the measuring nodes \( T^p_{j,c} \):

\[
Z(\bar{q}) = \sum_{p=1}^{M} \sum_{j=1}^{F} (T^p_{j,m} - T^p_{j,c} (\bar{q}))^2.
\]

(9)

No explicit regularization is applied in this algorithm. Instead, regularization is inherent in the choice of the approximation function taken for the unknown \( q \)'s. In the case where there are a large number of degrees of freedom, the above function may be regularized further by supplementing the objective function with, say, a first-order regularization term.

4. Solution by the LM method

When the LM method is adopted to solve the least-squares problem given in Eq. (9) then the following iteration is carried out to minimize \( Z(\bar{q}) \):

\[
\bar{q}^{k+1} = \bar{q}^k + [(J^k)^T J^k + \mu^k I]^{-1} J^k T^m (T - T (\bar{q}^k))
\]

(10)

where \( k \) denotes the iteration level, \( (J^k)^T \) is the matrix transpose of sensitivity coefficients \( J_{j,i} = \partial T_i / \partial q_j \), the matrix \( I \) is the identity matrix, and \( \mu^k \) is a positive scalar damping parameter whose magnitude is dynamically adjusted to regularize the iterative process. Both iterative convergence and the magnitude of the objective function are used as stopping criteria, i.e. the iteration is stopped when

\[
\| \bar{q}^{k+1} - \bar{q}^k \| \leq \varepsilon \quad \text{and} \quad Z(\bar{q}) \leq \delta,
\]

(11)

where \( \varepsilon \) and \( \delta \) are suitably chosen positive constants. Attention is now given to determining the sensitivity coefficients in an explicit manner which relies on the superposition principle and thus avoids differentiation with respect to the unknown heat fluxes. The standard IMSL subroutine DBCLSJ was used with default values of the parameters for \( \varepsilon, \delta, \) and \( \mu^k \).

5. Evaluation of the sensitivity coefficients

We present a general means to explicitly obtaining the sensitivities \( J_{j,i} \) required in the LM method. It is worthwhile noting that the method is applicable when using any field solver (say BEM or FEM for problems in arbitrary geometries which preclude analytical solutions or an
analytical solution for problems amenable to closed-form solutions). This development requires that the forward problem under consideration has to be linear. The collection of temperatures \( T_j \) at the \( j = 1, 2 \ldots M \) measuring nodes at the measured time levels \( p = 1, 2 \ldots F \) can always be written as

\[
T = \tilde{C} \tilde{\mathbf{q}} + \tilde{b},
\]

where \( T \) is \( [(M \times F) \times 1] \). The length of the vector of fluxes \( \tilde{\mathbf{q}} \) depends on the parametrization chosen to model the sought-after heat flux, and, it is in general of size \( [(N \times FI) \times 1] \). It is advantageous to keep \( N \) and \( FI \) as low as possible in order to take advantage of inherent smoothing (regularization) of the computed flux provided by the least-squares fit and in order to keep the number of parameters in the inverse problem low. The sensitivity matrix \( \tilde{C} \) is \( [(M \times F) \times (N \times FI)] \). The vector \( \tilde{b} \) is of size \( [(M \times F) \times 1] \). In this work, all components of Eq. (12) are computed using the convolution time-marching BEM. We now consider how to calculate the entries of \( \tilde{C} \) and \( \tilde{b} \).
Let us denote all boundary conditions arising from the problem at hand as

(a) heat fluxes to be retrieved,
(b) remaining boundary conditions, referred to as external.

We begin with the calculation of the \( b \) vector. Assume for a moment that all fluxes to be retrieved are zero while the external boundary conditions remain intact. The interpretation of the entries of the vector \( b \) are simply the solution of such a boundary value problem. Thus, the temperatures evaluated using the exact solution at sensor positions and setting all heat fluxes to be retrieved to zero are the sought-for components of the vector \( b \). These are computed by the BEM.

Interpretation of the columns of the \( C \) matrix is done as follows. Consider rendering all external boundary conditions homogeneous without changing their type. For instance at points where temperatures were prescribed, the temperature is now zero. Similarly, at points where the flux was prescribed, its value is set to zero. In the case of mixed boundary conditions of the type \( q = h(T - T_\infty) \) where \( h \) and \( T_\infty \) are known, the value of the latter is taken to be zero. Now, set the first heat flux in the vector \( \bar{q} \) for the first time approximating interval equal to 1 (meaning the prescribed flux is set to \( N_1(\rho) \)) while all other fluxes are zero, see the first row in Fig. 2. Using these boundary conditions, computed temperatures (analytical or numerical) at locations where the measurements are taken can then be easily recognized as being equal to the first column of the matrix \( C \). In a similar way all columns \( j \) of \( C \) are readily calculated. From definition of sensitivity coefficients, the following is immediately apparent, for a given index \( i \) as:

\[
J_{ij} = C_{ij}.
\]  

(13)

It should be noted that no access to the matrix itself is required in this approach, thus any code (BEM, FEM, FVM, etc.) or analytical solution, if one can be obtained to solve the direct problem, can be used to generate the sensitivity coefficients. Moreover, the coefficient matrix \( C \) has a certain structure with zero block elements located above the diagonal blocks, and this is a direct consequence of causality. Furthermore, when measurements are taken as periodic with respect to the time approximating interval \( \Delta t_f = t_f - t_{f-1} \), diagonal blocks can be shown to be repeating. These features can be capitalized upon to provide considerable computational savings. Formally, the assembled set has the structure shown in Fig. 3. Here, the submatrices \( C_{ij} \) are each \([M \times N]\) and the superscripts are used to indicate that each matrix is associated with a measuring time level \( p \) and a time interpolating interval \( f \).

It can be shown that

\[
C^{11} = C^{22} = C^{33} = \cdots = C^{FF},
\]

and that

\[
C^{31} = C^{32} = \cdots
\]

and so on. Thus, in practice, the full-matrix \( C \) is not assembled, only non-zero subsets are stored, and at every time approximating interval, the following sequence of problems is solved. Beginning with the first time approximating interval \( f = 1 \), the vector \( \bar{q}^1 \) is found from the solution to

\[
T_{m}^1 = C_{m}^{11} \bar{q}^1 + b_1.
\]  

(14)

This is then followed by the second time approximating interval \( f = 2 \), in which the second vector \( \bar{q}^2 \) is found from the solution to

\[
T_{m}^2 = C_{m}^{22} \bar{q}^2 + \left[ C_{m}^{21} \bar{q}^1 + b^2 \right]
\]

\[
= C_{m}^{22} \bar{q}^2 + b_2^2,
\]  

(15)

where

\[
\bar{b}_2 = C_{m}^{21} \bar{q}^1 + b^2
\]

since the vector \( \bar{q}^1 \) has been determined from the previous time approximation interval. The next problem is then at the third time approximating interval \( f = 3 \) at which the vector \( \bar{q}^3 \) is found from the solution to

\[
T_{m}^3 = C_{m}^{33} \bar{q}^3 + \left[ C_{m}^{31} \bar{q}^1 + C_{m}^{32} \bar{q}^2 + b_3 \right]
\]

\[
= C_{m}^{33} \bar{q}^3 + b_3^3,
\]  

(16)

where

\[
\bar{b}_3 = C_{m}^{31} \bar{q}^1 + C_{m}^{32} \bar{q}^2 + b_3
\]

have been determined from the previous time approximation levels. This process is continued until the final time.
approximating interval $f = FI$ is reached and the last vector $\vec{q}^F$ is found. This concludes the theoretical developments. Attention is now given to a numerical validation example.

6. Example

A numerical example is considered in which the heat flux is to be recovered over the suction side (top) of the airfoil, nodes $j = 21 - 39$ as illustrated in Fig. 4. Material properties, BEM discretization and boundary conditions are given in the figure. SI units are used throughout. The forward problem is solved using the BEM with the assumed flux illustrated by the solid lines in Fig. 5 until 90s using nine constant-in-time elements and 40 subparametric quadratic elements. The heat flux is then determined using the computed surface temperature histories at the BEM nodes at the suction side with and without simulated random error (maximum of $\pm 2K$) added to simulated TLC measurements. Seven roof functions, $N_j(r)$, are used to model the spatial distribution of the heat flux and three constant temporal shape functions, $N_f(t)$, are used to model the temporal variation of the heat flux. Results reveal that the heat flux is accurately retrieved both in space and time, including the cases in which random error was added to simulated input temperatures (Fig. 6).
7. Conclusions

In this paper, we present a BEM-based approach to reconstruct an unknown time dependent heat flux distribution at a surface whose temperature history is provided by a broad-band thermochromic liquid crystal (TLC) thermographic technique. The information given for this inverse problem is the surface temperature history. Although this is not an inverse problem, it is solved as such in order to filter the errors in input temperatures which are reflected in errors of heat fluxes. We minimize a quadratic functional which measures the sum of the squares of the deviation of estimated (computed) temperatures relative to those measured temperatures provided by the TLC thermography. The objective function is minimized using the LM method, and we develop an explicit scheme to compute the required sensitivity coefficients. The unknown flux is allowed to vary in space and time. Results are presented for a simulation in which a spatially varying and time-dependent flux is reconstructed over an airfoil, and these show that the proposed method is capable of faithfully reproducing the heat flux in space and time.

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References


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